# COUNTING LATTICE POINTS AND THE NUMBER $2i + 7$

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Abstract. Let $P$ be a lattice $d$-dimensional polytope. A theorem of E. Ehrhart states that the number of lattice points of $nP$ (the $n$-th dilate of $P$), as a function of positive integers $n$, agrees with a polynomial of degree $d$ (the Ehrhart polynomial of $P$). In this talk, I will give a review of Ehrhart polynomials and discuss a paper of C. Haase and J. Schicho that classifies all such polynomials for $d = 2$.

If we replace "lattice polytope" by "rational polytope," the word "polynomial" becomes "polynomial with periodic coefficients" (quasi-polynomials). I will discuss known bounds for the periods of Ehrhart quasi-polynomials, but show that in some cases, they are ordinary polynomials. I will also show that the classification for $d = 2$ is incomplete if we consider rational polygons whose Ehrhart quasi-polynomial is a polynomial, and present some conjectures that I like about why some polytopes have smaller period than the known bounds suggest.

An *integral* polygon in $\mathbb{R}^2$ is one whose vertices have integer coordinates. It need not be convex.

**Theorem 0.1** (Pick 1899)**.** *Let $P$ be an integral polygon in $\mathbb{R}^2$. Let $A = A(P)$ be the area of $P$. Let $I = I(P)$ be the number of interior lattice points. Let $B = B(P)$ be the number of points on the boundary of $P$. Then $A = I + B/2 - 1$.*

Let $t \in \mathbb{Z}_{>0}$. Let $tP = \{tx : x \in P\}$. Then $A(tP) = A(P)t^2$ and $B(tP) = B(P)t$. One finds that

$$\#(tP \cap \mathbb{Z}^2) = At^2 + \frac{B}{2}t + 1.$$

**Theorem 0.2** (Ehrhart 1962)**.** *Let $P \subset \mathbb{R}^n$ be an integral polytope. There exists a polynomial $L_P(t)$ such that $L_P(t) = \#(tP \cap \mathbb{Z}^n)$ for all $t \in \mathbb{Z}_{>0}$.*

Natural question: What polynomials are Ehrhart polynomials? If the dimension is 1, this is trivial: $at + 1$. If the dimension is 2, a classification is known. If the dimension is $\geq 3$, it is not known.

For $P \subset \mathbb{R}^n$, let $C$ be the cone generated by $P \times \{1\}$ in $\mathbb{R}^{n+1}$. The *Ehrhart ring* $R \subseteq k[x_1, \dots, x_{n+1}]$ is the $k$-span of $x_1^{m_1} \cdots x_{n+1}^{m_{n+1}} \in R$ for $(m_1, \dots, m_{n+1}) \in C \cap \mathbb{Z}^{n+1}$; give it the grading $\deg(x^m) = m_{n+1}$. The Ehrhart polynomial is the Hilbert polynomial of the Ehrhart ring.

The ring $R$ is Cohen-Macaulay, and $\dim R = \dim P + 1$. It need not be generated in degree 1: consider the convex hull $P$ of the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,2)$.

Connection to toric varieties: Given a polytope $P \subseteq M$ with $M := \mathrm{Hom}(N, \mathbb{Z})$, construct the normal fan $\Delta_P$ of $P$. For each face $Q$ of $P$, define $\sigma_Q = \{v \in N \otimes \mathbb{R} : \langle u, v\rangle \leq \langle u', v\rangle \text{ for all } u \in Q,\ u' \in P\}$. For each $\sigma \in \Delta$, let $S_\sigma := \{u \in M : \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}$ and let $U_\sigma = \operatorname{Spec} k[S_\sigma]$. For an inclusion $\tau \subseteq \sigma$, we get an open immersion $U_\tau \hookrightarrow U_\sigma$ which are compatible, so glue them. The result is $X_{\Delta_P} = \operatorname{Proj} R_P$.

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**Definition 0.3.** A *$T$-Cartier divisor* on $X_P$ is a Cartier divisor that comes from a Weil divisor $D = \sum a_i D_i$ such that $D_i$ corresponds to a $T$-invariant codimension 1 subvariety (which corresponds to a ray of $\Delta_P$).

Number the rays $\tau_1, \dots, \tau_r$. Let $v_i$ be the first nonzero lattice point on $\tau_i$. Then $D = \sum a_i D_i$ gives rise to the rational polyhedron

$$P_D := \{u \in M \otimes \mathbb{R} : \langle u_i, v_i \rangle \geq -a_i \text{ for all } i\}.$$

The upshot is that from $P$ we get a $T$-Cartier divisor $D$ on $X_P$, such that $\mathcal{O}(D)$ is ample. Then $L_P(t) = \chi(\mathcal{O}(D)^{\otimes t})$.

Back to classification: For $d = 2$, classify triples $(A(P), I(P), B(P)) \subset \mathbb{R}^3$. Pick: These points lie on a plane in $\mathbb{R}^3$. Obvious: $B(P) \geq 3$, $A(P) \geq 1/2$.

**Theorem 0.4.**

- *If $I = 1$, then $3 \leq B \leq 9$.*
- *If $I \geq 2$, then $3 < B \leq 2I + 6$.*
- *These, together with Pick and the obvious conditions, are the only conditions.*

Examples: The triangle with vertices $(0,0)$, $(b,0)$, and $(0,1)$ gives $i = 0$, $b \geq 3$. The triangle with vertices $(0,0)$, $(3,0)$, and $(0,3)$ gives $i = 1$, $b = 9$. The triangle with vertices $(0,0)$, $(4,0)$, and $(0,2)$ gives $i = 1$, $b = 8$. The quadrilateral with vertices $(0,0)$, $(b-4,0)$, $(i+1,1)$, and $(0,2)$ gives $i \geq 1$, $4 \leq b \leq 2i+5$. The rectangle with vertices $(0,0)$, $(i+2,0)$, $(i+2,2)$, $(0,2)$ gives $i \geq 1$, $b = 2i+6$.

## 1. Algebraic geometry

Let $X$ be a normal rational surface. Let $d = \deg X$. Let $p$ be the sectional genus of $X$. If $X = X_P$, $p = I(P)$, $d = 2A(P)$.

**Theorem 1.1** (Schicho 1999)**.** *If $p = 1$, then $d \leq 9$. If $p \geq 2$, then $d \leq 4(p+1)$.*

## 2. Onion skins

**Definition 2.1.** Given $P \subset \mathbb{R}^2$, define $P = P^{(0)} \supset P^{(1)} \supset P^{(2)} \supset \cdots$, where $P^{(n+1)}$ as the convex hull of the set of interior lattice points of $P^{(n)}$.

Let $\Delta$ be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

**Definition 2.2.** The *level* of $P$ is

$$\ell(P) := \begin{cases} n, & \text{if } P^{(n)} \text{ is a point or segment} \\ n + 1/3, & \text{if } P^{(n)} \text{ is equivalent to } \Delta \\ n + 2/3, & \text{if } P^{(n)} \text{ is equivalent to } 2\Delta \\ n + 1/2, & \text{otherwise, where } n \text{ is the last integer such that } P^{(n)} \text{ is nonempty.} \end{cases}$$

**Theorem 2.3** (Haase-Schicho)**.** *If $\ell = \ell(P) \geq 1$, then $(2\ell - 1)B \leq 2I + 9\ell^2 - 2$. This is a refinement.*

What is the level of a surface? Let

$$\ell(X) = \sup\{\ell \in \mathbb{Q} : H + \ell K \text{ is effective}\},$$

where $H$ is the divisor of a hyperplane section, and $K$ is a canonical divisor. Then

$$(2\ell - 1)d \leq 9\ell^2 + 4\ell(p-1)$$

for toric surfaces. Is this true for non-toric rational surfaces?

Reference: Christian Haase and Josef Schicho, "Lattice polygons and the number $2i + 7$".