THE ABSOLUTE INTEGRAL CLOSURE IN CHARACTERISTIC p (AN EXPOSITION OF WORK BY HOCHSTER, HUNEKE AND KNOP)

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Abstract. Let R be a local Noetherian domain of positive characteristic. A theorem of Hochster and Huneke shows that the absolute integral closure of R is Cohen-Macaulay if R is excellent. The existence of big Cohen-Macaulay algebras is one of the homological conjectures, and indeed rather a strong one; it implies, for instance, the monomial conjecture (if $x_1, \ldots, x_d$ is a system of parameters, then $(x_1, x_2, ..., x_d)^n$ is not in the ideal generated by the $(n+1)$-th powers of the x_i, for any n). I will present a simpler proof of this result, given by Huneke and Lyubeznik, which extends to the case where R is the image of a Gorenstein local ring, and elaborate somewhat on these connections.

Theorem 0.1. *If R is a noetherian domain that is the image of a Gorenstein local ring A of characteristic p, then the integral closure R^+ of R (in an algebraic closure of its fraction field) is CM (Cohen-Macaulay).*

This follows from a better theorem:

Theorem 0.2. *Let R be as above. For every R-subalgebras $R' \subset R^+$ that is finite over R, there exists an R'-subalgebra R'' of R^+ that is finite over R such that $H^i_{\mathfrak{m}}(R') \to H^i_{\mathfrak{m}}(R'')$ is 0 for all $i < d := \dim R$.*

Consequences:

(1) $H^i_{\mathfrak{m}}(R^+) = 0$ for all $i < d$.
(2) R^+ is Cohen-Macaulay: every system of parameters on R is regular on R^+.

Proof of (2). Assume that $(x_1, \ldots, x_{j-1})$ is regular. Let $I_t = (x_1, \ldots, x_t)$. For $t \leq j-1$, taking cohomology of the exact sequence

$$0 \to R^+/I_{t-1}R^+ \xrightarrow{x_t} R^+/I_{t-1} \to R^+/I_tR^+ \to 0$$

yields $H^q_{\mathfrak{m}}(R^+/I_{j-1}R^+) = 0$ for all $q < d-(j-1)$. Since $j-1 < d$, we get $H^0_{\mathfrak{m}}(R^+/I_{j-1}R^+) = 0$. □

Let A be Gorenstein with $A \twoheadrightarrow R$. Let $n = \dim A$. Local duality gives $H^i_{\mathfrak{m}}(-) \simeq D(\mathrm{Ext}^{n-i}_A(-, A))$ on finite R-modules, where $D = \mathrm{Hom}(-, E)$, where E is the injective hull of $A/\mathfrak{m}A$.

We use induction on $\dim R$. The module $N := \mathrm{Ext}^{n-i}_A(R', A)$ is finite over R. Let $\mathfrak{p}$ be a prime ideal not equal to $\mathfrak{m}$. Claim (*): There exists an R'-subalgebra $R^{\mathfrak{p}}$ of R^+, finite over R, such that for every $R^{\mathfrak{p}}$-algebra R^*, finite over R, the map $\mathrm{Ext}^{n-i}_A(R*, A) \to \mathrm{Ext}^{n-i}_A(R', A) =: N$ is zero when localized at $\mathfrak{p}$. It suffices to prove that $\mathrm{Ext}^{n-i}_A(R*, A) \to \mathrm{Ext}^{n-i}_A(R^{\mathfrak{p}}, A)$. Let $d_{\mathfrak{p}} = \dim R/\mathfrak{p} > 0$. Then $\dim R_{\mathfrak{p}} = d - d_{\mathfrak{p}} < d$. Since $i < d$, we have $i - d_{\mathfrak{p}} < d - d_{\mathfrak{p}}$, so by the inductive hypothesis, there exists $T_{\mathfrak{p}}$, finite over $R_{\mathfrak{p}}$, such that $H^i_{\mathfrak{p}}(R_{\mathfrak{p}}) \to H^i_{\mathfrak{p}}(T_{\mathfrak{p}})$

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is 0. Say $T_{\mathfrak{p}} = R_{\mathfrak{p}}[z_1, \dots, z_n]$. Without loss of generality the z_i are integral over R. Let $R^{\mathfrak{p}} := R[z_1, \dots, z_n]$.

Consider $\overline{R} := R'[R^{\mathfrak{p}_1}, \dots, R^{\mathfrak{p}_j}]$. Dualizing $\operatorname{Ext}_A^{n-i}(\overline{R}, A) \twoheadrightarrow \mathcal{I} \to \operatorname{Ext}_A^{n-i}(R', A)$ yields $H^i_{\mathfrak{m}}(R') \twoheadrightarrow D(\mathcal{I}) \hookrightarrow H^i_{\mathfrak{m}}(\overline{R})$; let ϕ be the composite map. The image $\operatorname{im}\phi$ is finitely generated, say by $\alpha_1, \dots, \alpha_t$. Also, $\operatorname{im}\phi$ is Frobenius-stable.

Lemma 0.3. *Let R be a commutative noetherian domain. Let $\alpha \in H^i_{\mathfrak{m}}(R)$ be such that the elements α^q for $q = p^e$ lie in a finitely generated submodule. Then there exists R' finite over R such that α maps to 0 under $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R')$.*

Proof. Suppose

$$g(\alpha) := \alpha^{p^s} - r_1\alpha^{p^{s-1}} - \cdots - r_{s-1}\alpha = 0.$$

Let $\tilde{\alpha} \in C^i$ be a Cech cocycle representing α. Then $g(\alpha) = 0$ implies $g(\tilde{\alpha}) = d\beta$ for some $\beta \in C_{i-1}$. For $\lambda = (1 \le j_1 < j_2 < \cdots < j_{i-1} \le d)$, write $\beta_\lambda = r_\lambda / x_\lambda$ where $x_\lambda = x_{j_1}^{e_1} \cdots x_{j_{i-1}}^{e_{i-1}}$. ☐

Consider $g(Z_\lambda/x_\lambda) - r_\lambda/x_\lambda$. Multiply by $(x_\lambda)^{p^s}$. This clears the denominators, and exactly clears the leading denominator. A root z_λ of it is integral. Adjoint all the z_λ. Without loss of generality, they were already in the ring.

Let $\tilde{\tilde{\alpha}} = \left(\frac{z_\lambda}{x_\lambda}\right) \in C_{i-1}$. Then $g(\tilde{\tilde{\alpha}} = \beta$, $g(\tilde{\alpha}) = d\beta$, and $\bar{\alpha} = \tilde{\alpha} - d\tilde{\tilde{\alpha}}$. Now $\bar{\alpha}_\lambda = \left(\frac{s_\lambda}{x_\lambda}\right)$, $g(Y_\lambda) = 0$, $g(\bar{\alpha}) = 0$.