FINITE GROUP REPRESENTATIONS AND LOCAL SUBGROUPS

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Abstract. There appear to be a number of unexplained (and unproved) connections between the representations of a finite group G and the representations of certain subgroups of G. Given a prime p, the normalizer in G of a nontrivial subgroup of p-power order is called a "p-local" subgroup of G. It is the representations of these p-local subgroups that seem to influence the representation theory of G. This talk will discuss some of the outstanding conjectures in this area, with an emphasis on the McKay conjecture and some of its variations.

1. p-local subgroups

Let G be a group (all groups will be finite). Let p be a prime.

Definition 1.1. Say that a subgroup $L \subseteq G$ is *p-local* if L is the normalizer $N_G(P)$ of some subgroup $P \subset G$ with $P > 1$ and $|P|$ a power of p.

These are the subgroups that "one can lay one's hands on". Note that the p-Sylow groups themselves may not be p-local according to this definition, though group theorists are sometimes sloppy about this, and call p-local anything that can be gotten from information from such subgroups.

The general principle is that "local implies global".

(1) (Frobenius) If every p-local has a normal p-complement, then G has a normal p-complement. (A normal p-complement is a normal subgroup C of G such that $|C|$ is prime to p, and $(G : C)$ is a power of p.) There is a strengthening due to Thompson: one need look at the normalizers of only two subgroups: of the center of a p-Sylow subgroup and of the Thompson subgroup.

(2) (Burnside) Let $e \in \mathbb{Z}_{\geq 0}$. Assume that a Sylow p-subgroup P of G is abelian. If $N_G(P)$ has a normal subgroup of index p^e, then G has one. (This can sometimes be used to prove that a group G is not simple.)

(3) If every p-local subgroup of G for every odd p has a normal Sylow 2-subgroup, then G has one.

2. Representation theory

How does "local implies global" arise in representation theory?

Let F be a field. A *representation of* G is a homomorphism

$$\mathcal{X} \colon G \to \mathrm{GL}(n, F).$$

The integer n is called the *degree* of the representation.

Assume for now that $F = \mathbb{C}$. Define the *character* of $\mathcal{X}$ by $\chi(g) := \operatorname{tr} \mathcal{X}(g)$ for $g \in G$. In particular, $\chi(1) = n$. The character determines $\mathcal{X}$ up to conjugation. Call χ *irreducible* if

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it is not a sum of two nonzero characters. Let $\operatorname{Irr}(G)$ be the set of irreducible characters of G. Then $|\operatorname{Irr}(G)|$ is the number of conjugacy classes of G: in fact, the irreducible characters form a basis for the vector space of functions $G \to \mathbb{C}$ that are constant on conjugacy classes in G. Also, $\chi(1)$ divides $|G|$ for $\chi \in \operatorname{Irr}(G)$. Moreover, $\sum_{\chi \in \operatorname{Irr}(G)} \chi(1)^2 = |G|$.

Remark 2.1. Suppose instead that F is an algebraically closed field of characteristic p. Then the divisibility and $\sum_{\chi \in \operatorname{Irr}(G)} (\deg \chi)^2 = |G|$ both fail. Brauer's theorem states that $|\operatorname{Irr}(G)|$ equals the number of conjugacy classes of G whose elements are of order not divisible by p.

Remark 2.2. The theory of Brauer characters, developed by Brauer in the 1930s, lets one get around some of these problems with character theory in characteristic p. These Brauer characters are defined on elements of order not divisible by p, and are complex-valued.

3. Alperin Weight Conjecture

For a positive integer n, let n_p be the highest power of p dividing n.

Definition 3.1. A *p-weight* (or simply *weight*) in G is an ordered pair (P, θ) where $P \subseteq G$ and $|P|$ is a power of p (possibly 1), and $\theta \in \operatorname{Irr}(N/P)$, where $N := N_G(P)$, and $\theta(1)_p = |N/P|_p$. (One says that θ is of p-defect zero.)

The group G acts via conjugation on the set of weights.

Conjecture 3.2 (Alperin Weight Conjecture)**.** The number of orbits of p-weights in G equals the number of conjugacy classes of G of elements of order not divisible by p.

Example 3.3. If p does not divide $|G|$, we recover the fact that the number of irreducible characters equals the number of conjugacy classes of G.

Example 3.4. Let $G = A_5$. Let $p = 2$. We have $|G| = 4 \cdot 15$.

$\lvert P\rvert$	$\lvert N\rvert$	$\lvert N/P\rvert$	degrees of characters of N/P
1	60	60	$1, 3, 3, \mathbf{4}, 5$
2	4	2	$1, 1$
4	12	3	$\mathbf{1}, \mathbf{1}, \mathbf{1}$

The boldface entries correspond to characters of p-defect zero: these are what is counted on the left hand side of the Alperin Weight Conjecture. On the other hand, the conjugacy classes of order prime to 2 in A_5 have orders $1, 3, 5, 5$ (there are two conjugacy classes of 5-cycles in A_5).

4. McKay Conjecture

Fix a prime p. Define

$$m(G) := \{\chi \in \operatorname{Irr}(G) : p \nmid \chi(1)\}.$$

Let P be a Sylow p-subgroup of G. Let $N = N_G(P)$.

Conjecture 4.1 (McKay Conjecture)**.** $m(G) = m(N)$.

Example 4.2. Let $G = A_6$. Let $p = 5$. The characters have degrees $\mathbf{1}, 5, 5, \mathbf{8}, \mathbf{8}, \mathbf{9}, 10$. Thus $m(A_6) = 4$. (This time we write the degrees not divisible by $p = 5$ in boldface.) The group N is a dihedral group of order 10, with characters of degrees $\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}$, so $m(N) = 4$ too.

One can reduce the McKay Conjecture to a stronger statement for simple groups.

Conjecture 4.3 (Isaacs,Navarro)**.** Suppose that $1 \leq k < p/2$. Let

$$m_k(G) := \{\chi \in \mathrm{Irr}(G) : \chi(1) \equiv \pm k \pmod{p}\}.$$

Then $m_k(G) = m_k(N)$.