FINITE GROUP REPRESENTATIONS AND LOCAL SUBGROUPS

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Abstract. There appear to be a number of unexplained (and unproved) connections between the representations of a finite group $G$ and the representations of certain subgroups of $G$. Given a prime $p$, the normalizer in $G$ of a nontrivial subgroup of $p$-power order is called a “$p$-local” subgroup of $G$. It is the representations of these $p$-local subgroups that seem to influence the representation theory of $G$. This talk will discuss some of the outstanding conjectures in this area, with an emphasis on the McKay conjecture and some of its variations.

1. $p$-LOCAL SUBGROUPS

Let $G$ be a group (all groups will be finite). Let $p$ be a prime.

Definition 1.1. Say that a subgroup $L \subseteq G$ is $p$-local if $L$ is the normalizer $N_G(P)$ of some subgroup $P \subset G$ with $P > 1$ and $|P|$ a power of $p$.

These are the subgroups that “one can lay one’s hands on”. Note that the $p$-Sylow groups themselves may not be $p$-local according to this definition, though group theorists are sometimes sloppy about this, and call $p$-local anything that can be gotten from information from such subgroups.

The general principle is that “local implies global”.

(1) (Frobenius) If every $p$-local has a normal $p$-complement, then $G$ has a normal $p$-complement. (A normal $p$-complement is a normal subgroup $C$ of $G$ such that $|C|$ is prime to $p$, and $(G : C)$ is a power of $p$.) There is a strengthening due to Thompson: one need look at the normalizers of only two subgroups: of the center of a $p$-Sylow subgroup and of the Thompson subgroup.

(2) (Burnside) Let $e \in \mathbb{Z}_{\geq 0}$. Assume that a Sylow $p$-subgroup $P$ of $G$ is abelian. If $N_G(P)$ has a normal subgroup of index $p^e$, then $G$ has one. (This can sometimes be used to prove that a group $G$ is not simple.)

(3) If every $p$-local subgroup of $G$ for every odd $p$ has a normal Sylow 2-subgroup, then $G$ has one.

2. REPRESENTATION THEORY

How does “local implies global” arise in representation theory?

Let $F$ be a field. A representation of $G$ is a homomorphism

$$\chi: G \to GL(n, F).$$

The integer $n$ is called the degree of the representation.

Assume for now that $F = \mathbb{C}$. Define the character of $\chi$ by $\chi(g) := \text{tr} \chi(g)$ for $g \in G$. In particular, $\chi(1) = n$. The character determines $\chi$ up to conjugation. Call $\chi$ irreducible if

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it is not a sum of two nonzero characters. Let $\text{Irr}(G)$ be the set of irreducible characters of $G$. Then $|\text{Irr}(G)|$ is the number of conjugacy classes of $G$: in fact, the irreducible characters form a basis for the vector space of functions $G \to \mathbb{C}$ that are constant on conjugacy classes in $G$. Also, $\chi(1)$ divides $|G|$ for $\chi \in \text{Irr}(G)$. Moreover, $\sum_{\chi \in \text{Irr}(G)} \chi(1)^2 = |G|$.

Remark 2.1. Suppose instead that $F$ is an algebraically closed field of characteristic $p$. Then the divisibility and $\sum_{\chi \in \text{Irr}(G)} (\deg \chi)^2 = |G|$ both fail. Brauer’s theorem states that $|\text{Irr}(G)|$ equals the number of conjugacy classes of $G$ whose elements are of order not divisible by $p$.

Remark 2.2. The theory of Brauer characters, developed by Brauer in the 1930s, lets one get around some of these problems with character theory in characteristic $p$. These Brauer characters are defined on elements of order not divisible by $p$, and are complex-valued.

3. Alperin Weight Conjecture

For a positive integer $n$, let $n_p$ be the highest power of $p$ dividing $n$.

Definition 3.1. A $p$-weight (or simply weight) in $G$ is an ordered pair $(P, \theta)$ where $P \subseteq G$ and $|P|$ is a power of $p$ (possibly 1), and $\theta \in \text{Irr}(N/P)$, where $N := N_G(P)$, and $\theta(1)_p = |N/P|_p$. (One says that $\theta$ is of $p$-defect zero.)

The group $G$ acts via conjugation on the set of weights.

Conjecture 3.2 (Alperin Weight Conjecture). The number of orbits of $p$-weights in $G$ equals the number of conjugacy classes of $G$ of elements of order not divisible by $p$.

Example 3.3. If $p$ does not divide $|G|$, we recover the fact that the number of irreducible characters equals the number of conjugacy classes of $G$.

Example 3.4. Let $G = A_5$. Let $p = 2$. We have $|G| = 4 \cdot 15$.

| $|P|$ | $|N|$ | $|N/P|$ | degrees of characters of $N/P$ |
|---|---|---|---|
| 1 | 60 | 60 | 1, 3, 3, 4, 5 |
| 2 | 4 | 2 | 1, 1 |
| 4 | 12 | 3 | 1, 1, 1 |

The boldface entries correspond to characters of $p$-defect zero: these are what is counted on the left hand side of the Alperin Weight Conjecture. On the other hand, the conjugacy classes of order prime to 2 in $A_5$ have orders 1, 3, 5, 5 (there are two conjugacy classes of 5-cycles in $A_5$).

4. McKay Conjecture

Fix a prime $p$. Define $m(G) := \{ \chi \in \text{Irr}(G) : p \nmid \chi(1) \}$. Let $P$ be a Sylow $p$-subgroup of $G$. Let $N = N_G(P)$.

Conjecture 4.1 (McKay Conjecture). $m(G) = m(N)$.

Example 4.2. Let $G = A_6$. Let $p = 5$. The characters have degrees $1, 5, 5, 8, 8, 9, 10$. Thus $m(A_6) = 4$. (This time we write the degrees not divisible by $p = 5$ in boldface.) The group $N$ is a dihedral group of order 10, with characters of degrees $1, 1, 2, 2$, so $m(N) = 4$ too.

One can reduce the McKay Conjecture to a stronger statement for simple groups.
Conjecture 4.3 (Isaacs,Navarro). Suppose that $1 \leq k < p/2$. Let
\[ m_k(G) := \{ \chi \in \text{Irr}(G) : \chi(1) \equiv \pm k \pmod{p} \}. \]
Then $m_k(G) = m_k(N)$. 