THETA DUALITIES ON MODULI SPACES OF SHEAVES ON K3 SURFACES

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Abstract. When studying spaces of sheaves in algebraic geometry, there are natural divisors to consider in products of two, in a sense complementary, moduli spaces of sheaves on curves or surfaces. Such divisors, which can be viewed as analogues of the classical theta divisors in the Jacobian of a curve, induce maps between spaces of sections of corresponding line bundles on the two moduli spaces. Remarkably, these maps are isomorphisms in the case of curves, and have been conjectured to be isomorphisms for some moduli spaces of sheaves on surfaces, a phenomenon known as strange (or theta) duality.

After explaining the problem in general, I will focus on the case of moduli of sheaves on a K3 surface. When smooth, these moduli spaces have an irreducible holomorphic symplectic structure, and the geometry is especially beautiful. I will explain the main features of this setup. I will then give concrete examples of the duality, providing evidence for the general statement in this setting. If time permits, I will also allude to the case of sheaves on abelian surfaces. The talk will partly be based on work in progress with Dragos Oprea, but will also be an introduction to the basic facts in this field.

Let S be a smooth projective surface over $\mathbb{C}$. Let H be a polarization. Let $K_{\text{top}}(S)$ be K-theory of topological vector bundles on S. Then $K_{\text{top}}(S) = \mathbb{Z} \oplus H^2(S,\mathbb{Z}) \oplus \mathbb{Z}$, with a vector bundle E being mapped to $(r(E), c_1(E), \chi(E))$. Let $K(S)$ be the Grothendieck group of coherent sheaves. To $v, w \in K_{\text{top}}(S)$, associate $\chi(v.w)$, where $v.w$ is the product in K-theory, given by the derived tensor product. For a fixed $v \in K_{\text{top}}(S)$, denote by $\mathcal{M}_v$ the Gieseker moduli space of H-semistable sheaves with topological type v. This is a projective scheme constructed by GIT (geometric invariant theory): $\mathcal{M}_v = \text{Quot} /\!/ \text{GL}(N)$.

Let $v^\perp \subseteq K(S)$ be the orthogonal complement of v. There is a group homomorphism

$$\theta\colon v^\perp \to \text{Pic}(\mathcal{M}_v).$$

If there exists a universal $\mathcal{E}$ over $\mathcal{M}_v \times S$ (which has projections p to $\mathcal{M}_v$ and q to S), and $w = [F]$, then $\det(Rp_*(\mathcal{E} \otimes q^*F))^{-1}$.

Pick v, w such that $\chi(v.w) = 0$. Look at $\mathcal{M}_v \times \mathcal{M}_w$.

Assumptions:

1. $H^2(E \otimes F) = 0$ for $(E,F) \in \mathcal{M}_v \times \mathcal{M}_w$ outside a locus of high codimension in the product. (This might seem very restrictive, but stability forces this in some cases, for instance, when $c_1(E \otimes F).H \gg 0$.)
2. The locus $\{(E,F) : h^0(E \otimes F) \neq 0\}$ gives a divisor $\Theta \subset \mathcal{M}_v \times \mathcal{M}_w$.
3. $\mathcal{O}(\Theta) = \theta_w \boxtimes \theta_v$ over $\mathcal{M}_v \times \mathcal{M}_w$. This is true if one of the factors is simply connected.

Each $\theta \in H^0(\mathcal{M}_v, \theta_w) \otimes H^0(\mathcal{M}_v, \theta_v)$ induces $D\colon H^0(\mathcal{M}_v, \theta_w)^\vee \to H^0(\mathcal{M}_w, \theta_v)$.

Two questions:

1. Is it the case that $\chi(\mathcal{M}_v, \theta_w) = \chi(\mathcal{M}_w, \theta_v)$?
2. If the higher cohomology vanishes, is D an isomorphism?

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These are speculations of Le Potier. See work of his student Gentiana Danila (2000): strange duality on $\mathbb{P}^2$ (limited number of cases).

Remark 0.1. It is not easy to calculate

$$\chi(\mathcal{M}_v, \theta_w) = \int e^{c_1(\theta_w)} \operatorname{todd}(\mathcal{M}_v).$$

These numbers can be viewed as modified Donaldson invariants.

Example 0.2. Let $S^{[k]}$ be the Hilbert scheme of k points on S. Let $\mathcal{I}_Z$ be the ideal sheaf of a length-k 0-dimensional subscheme Z of S. We have a universal sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O} \to \mathcal{O}_Z \to 0$$

over $S^{[k]} \times S$. For every sheaf F on S, define $F^{[k]} := \det(Rp_*(\mathcal{O}_Z \otimes q^*F))$. If L is a line bundle on S, then

$$H^0(S^{[k]}, L^{[k]}) = H^0(S \times \cdots \times S, L \times \cdots \times L)^{\Sigma_{k,a}} = \bigwedge^k(S, L),$$

(where each product has k factors). Pick a line bundle L on S such that $\chi(L) = n$ such that L has no higher cohomology. Let $v = [\mathcal{I}_Z]$ and $w = [\mathcal{I}_W \otimes L]$, where Z has length $k := \ell(Z) < n$. Then $\mathcal{M}_v = S^{[k]}$ and $\mathcal{M}_w = S^{[n-k]}$. Then

$$\Theta = \{(\mathcal{I}_Z, \mathcal{I}_W) : h^0(\mathcal{I}_Z \otimes \mathcal{I}_W \otimes L) \neq 0\},$$

and $\mathcal{O}(\Theta) = L^{[k]} \boxtimes L^{[n-k]}$, and we get an isomorphism

$$D \colon H^0(S^{[k]}, L^{[k]})^\vee \to H^0(S^{[n-k]}, L^{[n-k]})$$

$$\bigwedge^k H^0(S, L)^\vee \xrightarrow{\sim} \bigwedge^{n-k} H^0(S, L).$$

From now on, let S be a K3 surface. If E is a sheaf on S, we have the *Mukai vector* $v(E) := \operatorname{ch}(E)\sqrt{\operatorname{todd} S} \in H^{2*}(S, \mathbb{Z})$ and the *Mukai pairing*

$$\langle v, w \rangle = \int_S v_2 w_2 - v_0 w_4 - v_4 w_0.$$

Pick a Mukai vector v that is primitive in $H^{2*}(S, \mathbb{Z})$, with $v_0 > 0$. Also assume that the polarization H is generic. Then

(1) $\mathcal{M}_v$ is smooth projective, of dimension $n = \langle v, v \rangle + 2$, and consists only of stable sheaves (Mukai).
(2) $\mathcal{M}_v$ is an irreducible holomorphic symplectic variety (Kähler, simply connected, and $H^{2,0}$ is spanned by a nowhere degenerate holomorphic 2-form η) (Mukai).
(3) $\mathcal{M}_v$ is deformation equivalent to $S^{[\frac{1}{2}\langle v,v\rangle + 1]}$ (O'Grady and Yoshioka)
(4) There is an integral nondegenerate quadratic form B on $H^2(\mathcal{M}_v, \mathbb{Z})$ (the so-called Beauville-Bogomolov form) such that there is an isometry

$$\theta_v \colon H^{2*}(S, \mathbb{Z}) \supset v^\perp \xrightarrow{\sim} H^2(\mathcal{M}_v, \mathbb{Z});$$

i.e., the Mukai pairing corresponds to B (O'Grady and Yoshioka). Definition of B: If $\alpha \in H^2(\mathcal{M}_v, \mathbb{C})$ has Hodge decomposition $\lambda\eta + \beta + \mu\bar{\eta}$ where $\lambda, \mu \in \mathbb{C}$, then

$$B(\alpha) = \frac{n}{2} \int (\eta\bar{\eta})^{n-1}\beta^2 + \lambda\mu.$$

(5) $\chi(\mathcal{M}_v, \mathcal{L})$ is a deformation-invariant polynomial in $B(c_1(\mathcal{L}))$ (Fujiki). Therefore $\chi(\mathcal{M}_v, \theta_w) = \binom{d_v+d_w}{d_v}$ where $d_v := \frac{1}{2}\langle v, v\rangle + 1$ (sketched in O'Grady). In particular, $\chi(\mathcal{M}_v, \theta_w) = \chi(\mathcal{M}_w, \theta_v)$ holds in this case. Similar calculations work for an abelian surface instead of a K3 surface.

Suppose that S is an elliptic K3 surface with a section σ, i.e., there is a morphism $S \to \mathbb{P}^1$ and the general fiber is an elliptic curve. Suppose that $\mathrm{Pic}(S) = \mathbb{Z}\sigma + \mathbb{Z}f$. Let $\mathcal{M}_r^a$ be the moduli space of rank r sheaves E with $c_1(E).f = 1$ such that $\mathcal{M}_r^a$ has dimension $2a$. This determines E up to twisting $\mathcal{O}(f)$. Then $\chi(E_r) = 1$ at a generic point of this moduli space. Starting with the section given by $h^0(E_r) = \chi(E_r) = 1$, we get

$$0 \to \mathcal{O} \to E_r \to E_{r-1}(-2f) \to 0,$$

and $E_r \mapsto E_{r-1}$ gives a birational map $\mathcal{M}_r^a \dashrightarrow \mathcal{M}_{r-1}^a$ and so on until we reach $\mathcal{M}_1^a = S^{[a]}$. Thus $\mathcal{M}_r^a$ is birational to $S^{[a]}$. For some ν, let

$$\Theta = \{h^0(E_r \otimes F_s(\nu f)) \neq 0\} \subset \mathcal{M}_r^a \times \mathcal{M}_s^b.$$

Then $\mathcal{O}(\Theta) = L^{[a]} \boxtimes L^{[b]}$.

Question 0.3. Is $\Theta = \{h^0(\mathcal{I}_Z \otimes \mathcal{I}_W \otimes L) \neq 0\}$?

Almost certainly yes!