HYPERGRAPHIC DIVISORS, CURVES, AND MORPHISMS

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ABSTRACT. The birational geometry of the Grothendieck-Knudson moduli space of stable rational curves is still largely a mystery. It is not known if the cone of effective divisors is polyhedral, if the cone of effective curves is generated by 1-strata (Fultons conjecture), if it has only finitely many small modifications, and so on. In a joint work with Ana-Maria Castravet, we found out that some of this complexity can be explained by the fact that this moduli space can be interpreted as the Brill-Noether locus of a bunch of very special reducible curves related to hypergraphs. Using this we constructed myriads of new extremal divisors and rigid curves.

Let $\overline{M}_{0,n}$ be the moduli space of stable rational curves with n marked points. It is a smooth projective variety. The rank of $\operatorname{Pic}\overline{M}_{0,n}$ is something like 2^n. For each subset $I \subseteq \{1, 2, \ldots, n\}$, one has a divisor $\delta_I \hookrightarrow M_{0,n}$.

Problem: Find generators of $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$.

Same question for the Mori cone $\overline{\mathrm{NE}}_1 \hookrightarrow \mathrm{NS}^*_{\mathbb{R}}(\overline{M}_{0,n})$.

Speculations:

(1) Do natural strata generate the cones of effective cycles in any dimension? (Fulton)
(2) Is $\overline{M}_{0,n}$ a Mori dream space? (Keel-Hu) In $\overline{\mathrm{Eff}}$ one has the ample cone, and its closure, which is the nef cone. If $\overline{M}_{0,n} \dashrightarrow X'$ is a birational modification that is an isomorphism in codimension 1, one also has the nef cone of X'. Each $\mathrm{Nef}(X')$ has finitely many generators that are semiample. These chambers form $\mathrm{Mov}(\overline{M}_{0,n})$. Mori dream space means that $\mathrm{Mov}(\overline{M}_{0,n})$ is a finite union of chambers.

Example 0.1. Keel, Vermeire: $\mathrm{Eff}(\overline{M}_{0,6})$ is not generated by boundary divisors δ_I.

One has $\overline{M}_{0,6} \xrightarrow{f} \overline{M}_3$, obtained by gluing the six points in pairs to get a nodal genus-3 curve. The pullback $f^*\mathcal{H}_3$ of the hyperelliptic locus $\mathcal{H}_3$ in $\overline{M}_3$ gives a new divisor on $\overline{M}_{0,6}$.

Hassett-Tschinkel: $\overline{\mathrm{Eff}}$ is generated by δ_I's and the Keel-Vermeire divisor.

The proofs are by calculations in $\mathrm{Pic}(\overline{M}_{0,6})$.

In fact $\overline{M}_{0,6})$ is a Mori dream space. One way to see this: $\overline{M}_{0,6}$ is log Fano, and any log Fano variety is a Mori dream space. Alternatively, construct $\bigoplus_{L\in \mathrm{Pic}(\overline{M}_{0,6})} H^0(\overline{M}_{0,6}, L)$, and show that it is generated by δ_I's and the Keel-Vermeire divisor.

Keel-McKernan: $\overline{\mathrm{NE}}_1$ is generated by 1-strata.

Example 0.2. More generally, one can consider $\overline{M}_{0,2n} \to \overline{M}_n$ and pull back gonality divisors (the curves with gonality one less than expected), Brill-Noether divisors, Koszul divisors (Farkas), ...

First goal: find new extremal divisors (not pullbacks for forgetful maps for $\overline{M}_{0,n} \to \overline{M}_{0,k}$), to find generators for $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$, with a *geometric* proof of extremality.

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Gibney's Ph.D. thesis: If $g > 0$ and $\mathcal{M}_{g,n} \to Z$ is any birational morphism, then the exceptional locus is contained in $\partial \overline{M}_{g,n} := \overline{M}_{g,n} - M_{g,n}$.

The same techniques were used later by Gibney, Keel, and Morrison to prove that Fulton's conjecture about 1-cycles on $\overline{M}_{g,n}$ follows from Fulton's conjecture on $\overline{M}_{0,N}$.

Second goal: Give examples of birational morphisms $\overline{M}0, n \to Z$ with exceptional loci intersecting $M_{0,n}$ in all possible dimensions.

Third goal: Find rigid curves on $\overline{M}_{0,n}$.

Definition 0.3. A curve $C \hookrightarrow X$ *moves* if there exists a surface $S \xrightarrow{f} X$ with a proper morphism to a curve B such that $\dim f(S) = 2$ and $f(F) = C$ (set-theoretically) for some fiber F of $S \to B$.

Theorem 0.4 (Keel-McKernan)**.** *Let X be a $\mathbb{Q}$-factorial projective variety. Let $D \hookrightarrow X$ be a (possibly reducible) divisor with an ample support and with all components of D having anti-nef normal bundles $\mathcal{O}(D_i)|_{D_i}$ and let $C \hookrightarrow X$ be a curve that moves, and suppose that the ray spanned by C generates an edge of $\overline{\mathrm{NE}}_1(X)$. Then R also is spanned by a curve in D.*

Example 0.5. Let $X = \overline{M}_{0,n}$ and $D = \partial \overline{M}_{0,n}$.

Construction: A *hypergraph* is a collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_\ell\}$ where each "hyperedge" Γ_k is a subset of $\{1, 2, \ldots, n\}$. The elements of $\{1, 2, \ldots, n\}$ are called vertices. Suppose that

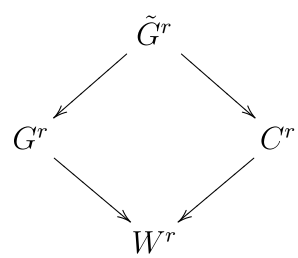
- $|\Gamma_k| \geq 3$ for each k.
- Γ is connected.
- Each vertex belongs to at least 2 hyperedges.

Let Σ be a reducible curve with one component isomorphic to $\mathbb{P}^1$ for each triple contained in each Γ_k; the triple is identified with three points of that $\mathbb{P}^1$, and these points are glued to the corresponding points of the other $\mathbb{P}^1$'s labelled by the same element of Γ.

Favorite: $\Gamma = 123, 145, 246, 356$.

Let's define the Brill-Noether loci. We have $\mathrm{Pic}^{1,1,\ldots,1}(\Sigma) \simeq \mathbb{G}_m^g$.

Let $W^r := \{L \in \mathbb{G}_m^g : h^0(L) \geq r+1\}$. Let $C^r := \{D \in \Sigma^r : \mathcal{O}(D) \in W^r\}$. Let $G^r := \{L \in W^r \text{ with } \mathbb{P}^1 \hookrightarrow \mathbb{P}(H^0(L))\}$. Let $\tilde{G}^r = \{L, \mathbb{P}^1, D : L \in W^r, D \in \mathbb{P}^1 \hookrightarrow \mathbb{P}(H^0(L))\}$.



Definition 0.6. Call D *admissible* if $D \cap \mathrm{Sing}\,\Sigma = \emptyset$. Call L *admissible* if it is basepoint-free and separates singular points.

Everything should be admissible in the diagram of loci above.

Let X be one of $W, C, G, \tilde{G}$. So $X^r \supset X^{r+1} \supset \cdots$. Suppose that $W^r \neq W^{r+1}$. Let $X_0^r = X^r - X^{r+1}$.

Let's find the fibers of these forgetful maps. The fibers of $\tilde{G}_0^r \to G_0^r$ are open subsets of $\mathbb{P}^1$. The fibers of $\tilde{G}_0^r \to C_0^r$ are open subsets of $\mathbb{P}^{r-1}$. The fibers of $C_0^r \to W_0^r$ are open subsets of $\mathbb{P}^r$. The fibers of $G_0^r \to W_0^r$ are open subsets in the Grassmannian $\mathrm{Gr}(2, r+1)$.

If $W^1 \neq W^2$, then $\tilde{G}^1 \to C^1$ is birational with an exceptional locus $\tilde{G}^2$.

We have the following observation:

Theorem 0.7. *We have* $\tilde{G}_1 \simeq M_{0,n+1}$, $M_{0,n} \simeq G^1$, $C^1 \simeq \prod_{\Gamma_k \in \Gamma} M_{0,\Gamma_k \cup \{n+1\}}$, *and* $W^1 = \prod_{\Gamma_k \in \Gamma} M_{0,\Gamma_k}$. *The map* $\tilde{G}^1 \to G^1$ *is "drop* $n+1$*". The map* $G^1 \to W^1$ *is the product of forgetful maps. The map* $C^1 \to W^1$ *is the forgetful map. The map* $\tilde{G}^1 \to C^1$ *is the hypergraphic morphism, again a product of forgetful maps.*

If $W^1(\Sigma) \neq W^2(\Sigma)$; i.e., there exists $\Sigma \to \mathbb{P}^2$ such that the image is a union of lines, but not a single line. The hypergraphic morphism $\overline{M}_{0,n+1} \to \prod_{\Gamma_k \in \Gamma} M_{0,\Gamma_k \cup \{n+1\}}$ contains $\tilde{G}^2$ in its exceptional locus.

Theorem 0.8. *The map* $\pi_\Gamma \colon \overline{M}_{0,n+1} \to \prod \overline{M}_{0,\Gamma_k \cup \{n+1\}}$ *is dominant if and only if for all* $S \subset \{1, \dots, \ell\}$,

$$\left| \bigcup_{i \in S} \Gamma_i \right| - 2 \geq \sum_{i \in S} \left(|\Gamma_i| - 2 \right).$$

And π_Γ *is birational if and only if* π_Γ *is dominant and* $n - 2 = \sum_{i=1}^{\ell} (|\Gamma_i| - 2)$.

Corollary 0.9. *Suppose that* π_Γ *is dominant and* $n - 2 = \sum_{i=1}^{\ell} (|\Gamma_i| - 2)$. *Suppose also that* $W^2(\Sigma) \neq W^1(\Sigma)$ *(*Σ *has a planar realization). Then* π_Γ *is a birational morphism, its exceptional locus is divisorial, nonempty on* $M_{0,n+1}$, *its components are some generators of* $\overline{\mathrm{Eff}}(M_{0,n+1})$ *and in fact edges, and their images in* $\overline{M}_{0,n}$ *are divisors, generators of* $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$.

Our favorite hypergraph gives a divisor in $\overline{M}_{0,7}$ contractible by a birational morphism, and its image in $\overline{M}_{0,6}$ is the Keel-Vermeire divisor.

Theorem 0.10. *If* Γ *consists of* $n-2$ *triples and*

$$\left| \bigcup_{i \in S} \right| \geq |S| + 3$$

for all S *such that* $2 \leq |S| \leq n - 3$, *then these divisors are new, i.e., not pullbacks.*

There is a "Fibonacci recursion" giving a sequence of hypergraphs (starting from our favorite hypergraph) and a sequence of new divisors on $\overline{M}_{0,7}$, $\overline{M}_{0,8}$, ...