HOMOLOGICAL DIMENSION AND THE CONTINUUM HYPOTHESIS (AFTER OSOFSKY)

Reference: B. Osofsky, Homological dimension and the Continuum Hypothesis.

Let R_n be $\mathbb{R}[x_1, \ldots, x_n]$ or $\mathbb{C}[x_1, \ldots, x_n]$. Let Q_n be its quotient field.

What can we say about the projective dimension $\operatorname{pd}_{R_n} Q_n$?

The Continuum Hypothesis will arise through the fact that $|\mathbb{R}| = 2^{\aleph_0}$.

1. Definitions and notation

Let R be a ring.

A directed R-module is a triple (M, M', u) such that M is an R-module, M' is a set of R-module generators of M, and $u\colon M' \times M' \to M'$ is a function (the "upper bound map") such that

(1) for all $x \in M'$ and $r \in R$, if $xr = 0$ then $r = 0$.
(2) For all $x, y \in M'$, we have $u(x, y)R \supset xR + yR$.

For $x, y \in M'$ define $x \leq y$ if $xR \subset yR$. Define $x > y$ to mean $xR \supsetneq yR$.

For $X \subset M$, let $P_n(X) = \bigoplus_{\substack{x_0 > x_1 > \ldots > x_n \\ \{x_0, \ldots, x_n\} \subset X}} \langle x_0, \ldots, x_n \rangle R$. Let $P_{-1}(X)$ be the submodule of M generated by X. For $x \in M'$, define $s(x) = \{y \in M' : y < x\}$ and $\bar{s}(x) = \{y \in M' : y \leq x\}$. Define

$$\begin{aligned} x^*\colon P_n(s(x)) &\to P_{n+1}(\bar{s}(x)) \\ \langle x_0, \ldots, x_n \rangle &\mapsto \langle x, x_0, \ldots, x_n \rangle. \end{aligned}$$

For $n = -1$, $x^*(xr) = \langle x \rangle r$.

Define $d_n\colon P_n(X) \to P_{n-1}(X)$ as follows. Let $d_0(\langle y \rangle) = y$. Let

$$d_n \langle x_0, \ldots, x_n \rangle = \sum_{i=0}^{n-1} (-1)^i \langle x_0, \ldots, \widehat{x_i}, \ldots, x_n \rangle + \langle x_0, \ldots, x_{n-1} \rangle (-1)^n x_{n-1}^{-1}(x_n).$$

Let $d_{n+1}(x^* p) = p - x^* d_n p$ for $n \geq 0$ and $p \in P_n(s(x))$.

2. Preliminaries

Proposition 2.1. *Let (M, M', u) be a directed R-module. Let $X \subseteq M'$ be u-closed; i.e., $u(X \times X) \subset X$. Then*

$$\cdots \xrightarrow{d_{n+1}} P_n(X) \xrightarrow{d_n} P_{n-1}(X) \to \cdots \to P_0(X) \xrightarrow{d_0} P_{-1}(X) \to 0$$

is a projective resolution of $P_{-1}(X)$.

Proposition 2.2. *Let $X \subset Y \subset M'$ be u-closed. Then*

$$P_n(X) \oplus P_{n+1}(Y) \xrightarrow{(-d_n, \mathrm{id}) \oplus d_{n+1}} \cdots \to P_0(X) \oplus P_0(Y) \xrightarrow{\mathrm{id} \oplus d_1} P_0(Y) \xrightarrow{d_0} \frac{P_{-1}(Y)}{P_{-1}(X)} \to 0$$

is a projective resolution.

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Theorem 2.3. *Let (M, M', u) be a directed R-module. Suppose that $|M'| = \aleph_n$ for some natural number n. Then $\mathrm{pd}_R(M) \leq n+1$.*

Lemma 2.4 (Auslander, "On the dimension of modules and algebras")**.** *Let I be a well-ordered set. Suppose that $\{N_i : i \in I\}$ is a collection of submodules of an R-module M such that $i \leq j$ implies $N_i \subset N_j$, and $M = \bigcup_{i \in I} N_i$. Suppose that $\mathrm{pd}_R\left(N_i / \bigcup_{j<i} N_j\right) \leq n$ for all $i \in I$. Then $\mathrm{pd}_R(M) \leq n$.*

3. The main proof

Proof of theorem. We use induction on n.

Suppose that $n = 0$. Then $M = \bigcup_{i \in \mathbb{N}} x_i R$ where $i \leq j$ implies $x_i R \subseteq x_j R$. For each i, the sequence

$$0 \to x_i R \to x_{i+1} R \to \frac{x_{i+1} R}{x_i R} \to 0$$

is a projective resolution of $\frac{x_{i+1}R}{x_i R}$, so $\mathrm{pd}_R\left(\frac{x_{i+1}R}{x_i R}\right) \leq 1$. By the lemma, $\mathrm{pd}_R(M) \leq 1$.

Now for the inductive step. Suppose that $|M'| = \aleph_{n+1}$. View $\aleph_{n+1}$ as an ordinal (the least ordinal having cardinality equal to $\aleph_{n+1}$). Label the elements of M': let $M' = \{x_\alpha | \alpha < \aleph_{n+1}\}$. Given any $Y \subset M'$, let $Y_0 = Y$, and let $Y_{n+1} = Y_n \cup u(Y_n \times Y_n)$, and define the closure $\mathrm{cl}(Y) = \bigcup_{n<\omega} Y_n$, so $\mathrm{cl}(Y)$ is the smallest u-closed subset of M' containing Y. Let $X_0 = \mathrm{cl}\{x_n : n < \omega\}$. Let $X_{\alpha+1} := \mathrm{cl}(X_\alpha \cup \{x_\alpha\})$. If λ is a limit ordinal, let $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$. Then $M' = \bigcup_{\alpha<\aleph_{n+1}} X_\alpha$ and $M = \bigcup_{\alpha<\aleph_{n+1}} P_{-1}(X_\alpha)$. For all $\alpha < \aleph_{n+1}$, we have $|X_\alpha| < \aleph_{n+1}$.

$$0 \to P_{-1}(\bigcup_{\beta<\alpha} X_\beta) \to P_{-1}(\bigcup_{\beta<\alpha} X_\beta) \to \frac{P_{-1}(X_\alpha)}{P_{-1}(\bigcup_{\beta<\alpha} X_\beta)} \to 0.$$

If $\alpha = \gamma + 1$, then $\bigcup_{\beta<\alpha} X_\beta = X_\gamma$. If instead α is a limit cardinal, then $\bigcup_{\beta<\alpha} X_\beta = X_\alpha$. In either case, we get

$$\mathrm{pd}_R\left(\frac{P_{-1}(X_\alpha)}{P_{-1}(\bigcup_{\beta<\alpha} X_\beta)}\right) \leq n+1.$$

Thus the lemma implies $\mathrm{pd}_R(M) \leq n+1$. □

4. Another result

Theorem 4.1. *Let R be a regular local ring (so in particular, R is a domain). Let $J(R)$ be the maximal ideal of R (i.e., Jacobson radical). Suppose that $|R/J(R)| = \aleph_k$ for some natural number k. Let Q be the quotient field of R. Suppose that the set of R-module generators of Q has size $\leq \aleph_k$. Then $\mathrm{pd}_R(Q) = \min\{n, k+1\}$.*

Corollary 4.2. *Let $R_n = \mathbb{C}[x_1, \ldots, x_n]$ and its fraction field Q_n. If $2^{\aleph_0} = \aleph_\ell$, then $\mathrm{pd}_{R_n}(Q_n) = \min(n, \ell+1)$.*

Proof. Let R^* be the localization of R_n at $(x_1, \ldots, x_n)$, so $|R^*| = |R_n| = |Q_n| = 2^{\aleph_0}$. The minimal set of R^*-module generators of Q has size equal to that of $R^*/J(R^*)$. Now Theorem 4.1 yields $\min(n, \ell+1) = \mathrm{pd}_{R*}(Q_n) \leq \mathrm{pd}_{R_n}(Q_n)$.

On the other hand, $\mathrm{pd}_{R_n}(Q_n) \leq n$ and the Theorem 2.3 yields $\mathrm{pd}_{R_n}(Q_n) \leq \ell + 1$. Thus $\mathrm{pd}_{R_n}(Q_n) \leq \min(n, \ell+1)$. □

Since ZFC allows for the possibility $2^{\aleph_0} = \aleph_\ell$ for any ℓ, then the answer to the question "What is $\mathrm{pd}_{R_n}(Q_n)$?" is not determined just by ZFC.