MODULES OF CONSTANT JORDAN TYPE

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Abstract. Let k be an algebraically closed field of characteristic p and let R be the ring $k[x_1, ..., x_n]/(x_1^p, ..., x_n^p)$. A finitely generated R-module is said to have constant Jordan type if the Jordan canonical form of $\lambda_1 x_1 + \cdots + \lambda_n x_n$ is independent of $\lambda_1, \ldots, \lambda_n$ (provided theyre not all zero). This concept was introduced and investigated in a recent paper of Carlson, Friedlander and Pevtsova. I'll describe some examples and some interesting properties of modules of constant Jordan type. I'll explain a theorem I proved a few weeks ago and a conjecture formulated by Rickard even more recently.

Let C denote J. Carlson. Let F denote E. Friedlander. Let P denote J. Pevtsova. Let S denote A. Suslin.

Let k be an algebraically closed field of characteristic p. Let E be an elementary abelian group, say $E \simeq (\mathbb{Z}/p)^r = \langle g_1, \ldots, g_r \rangle$. Let kE be the group algebra.

Let $X_i = g_i - 1$. Then $X_i^p = 0$. So $kE = k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$. If $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$, set $X_\alpha := \lambda_1 X_1 + \cdots + \lambda_r X_r \in J(kE)$, where $J(kE)$ is the Jacobson radical of kE. Then $X_\alpha^p = 0$.

If M is a finitely generated kE-module, we can view M as a $k[X_\alpha]$-module for some $\alpha \neq 0$; it breaks as a sum of Jordan blocks of eigenvalue 0 and length between 1 and p. The type will be denoted

$$[p]^{a_p}[p-1]^{a_{p-1}} \cdots [1]^{a_1}.$$

Warning: Even if $x - y \in J^2$, the elements x and y can have different Jordan canonical forms (Jcfs).

Nilpotent Jcfs are partially ordered: write $X \geq Y$ if for all $s > 0$, we have $\operatorname{rank} X^s \geq \operatorname{rank} Y^s$. Say that $x \in J(kG)$ has *maximal J type* if max with respect to this partial order.

Theorem 1 (FPS 2007)**.**

1. *If $x - y \in J^2(kE)$, then x has maximal type if and only if y has maximal type.*
2. *The elements of J/J^2 of maximal type form a dense open subset.*
3. *This agrees with the J type of the generic point of $\mathbb{A}^r(k)$.*

Definition 2. The *stable Jordan type* is

$$[p-1]^{a_{p-1}} \cdots [1]^{a_1}.$$

Definition 3 (CFP 2008)**.** Say that M has *constant Jordan type* if every element of $J - J^2$ has the same J type.

Question 4. If $r \geq 2$, what stable constant types can occur?

Example 5. The trivial module has scJt [1].

Example 6. The module $kE/J^2(kE)$ has scJt $[2][1]^{r-1}$.

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Lemma 7 (Dade 1978)**.** *If M has constant Jordan type $[p]^n$ for some n, then M is a free module. In particular, $p^{r-1}|n$.*

Theorem 8 (CFP)**.** *Every direct summand of a module of cJt has cJT.*

Tensor products: $M \otimes_k N$ is a module via $g(m \otimes n) = gm \otimes gn$.

Warning: The restriction of $M \otimes_k N$ to a $k[X_\alpha]$-module need not equal the tensor product of the restrictions of M and N.

Theorem 9 (CFP)**.** *If M, N have cJt, then $M \otimes_k N$ and $M^* := \mathrm{Hom}(M, k)$ have cJt too.*

Example 10. What modules M have scJt [1]? Then $M \otimes_k M^*$ also has scJt [1]. Consider the maps $k \to M \otimes_k M^* \to k$ given by $1 \mapsto \sum m_i \otimes m_i^*$ and $m \otimes f \to f(m)$. These split each other: $M \otimes_k M^*$ is the direct sum of k and a module of cJt $[p]$, and the latter summand is free by Dade's theorem. Call M an *endotrivial module* if $M \otimes_k M^*$ is the direct sum of k and a free module. Such modules have been classified for kE by Dade: $\Omega^n k$ for $n \in \mathbb{Z}$:

$$0 \to \Omega M \to P_M \to M \to 0$$

$$0 \to M \to I_M \to \Omega^{-1} M \to 0.$$

Theorem 11 (B. 2008)**.** *If $r \geq 2$, there is no module of scJt $[a]$ for $2 \leq a \leq p-2$.*

Proof. Write $\dim M = np + a$. The module $\Lambda^{a+1} M$ is projective and has cJt $[p]$, so it is free by Dade's theorem. Thus p^{r-1} divides

$$\dim \Lambda^{a+1} M = \frac{(np+a)(np+a-1)\cdots(np)}{1 \cdot 2 \cdots (a+1)},$$

so $p|n$. On the other hand,

$$\dim S^{p-a+1} M = \frac{(np+a)(np+a+1)\cdots(np+p)}{1 \cdot 2 \cdots (p-a+1)},$$

and by the following lemma with Dade's theorem, we get $p|(n+1)$. The previous two sentences give a contradiction. ☐

Lemma 12. *If $a + i > p$, then $S^i[a]$ is of type $[p]^?$.*

Proof. It's true if $a = p$; now use downwards induction on a. We have

$$0 \to S^i[a] \to S^{i+1}[a] \to S^{i+1}[a-1] \to 0.$$

Projective modules are also injective, so this completes the proof of the lemma. ☐

Question 13. Is there a module of scJt [3][1]?

Conjecture 14 (Rickard, MSRI 2008)**.** For $r \geq 2$, if there are no Jordan blocks of size $[i]$, then the number of J blocks of size $> i$ is divisible by p.

Vector bundles: locally free sheaves of $\mathcal{O}$-modules, where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^{r-1}}$.

Remark 15. It's not known whether there is an indecomposable rank 2 vector bundle on $\mathbb{P}^6$ or even on $\mathbb{P}^5$ (characteristic $\neq 2$). On $\mathbb{P}^4$, essentially the only ones known are the Horrocks-Mumford bundle $\mathcal{F}_{\mathrm{HM}}$ and others built from it in trivial ways.

Let $\mathbb{P}^{r-1} = \mathrm{Proj}\, k[Y_1, \ldots, Y_r]$, where Y_i is the function on $\mathbb{A}^r$ with $Y_i(X_j) = \delta_{ij}$.

Given a kE-module M, set $\tilde{M} = M \otimes_k \mathcal{O}$.

Definition 16 (FP 2007 preprint)**.** There is an operator $\theta\colon \tilde{M}(j) \to \tilde{M}(j+1)$ given by $\theta(m \otimes f) = \sum X_i m \otimes Y_i f$.

Definition 17 (BP 2008)**.** The module

$$\mathcal{F}_i(M) := \frac{\ker\theta \cap \operatorname{im}\theta^{i-1}}{\ker\theta \cap \operatorname{im}\theta^{i}}$$

as a subquotient of $\tilde{M}$.

Example 18. Let $M = kE/J^2(kE)$. Then M has cJt $[2][1]^{r-1}$. We have $\mathcal{F}_1(M) = \mathcal{T}(-1)$ and $\mathcal{F}_2(M) = \mathcal{O}(-1)$.

Example 19. Let $M = \operatorname{Soc}^2(kE)$. Then $\mathcal{F}_1(M) = \Omega(1)$, where Ω is the cotangent bundle, and $\mathcal{F}_2(M) = \mathcal{O}$.

Example 20 (B 2008)**.** If $p \geq 7$ and $r = 5$, then there exists a kE-module M such that $\mathcal{F}_2(M) \simeq \mathcal{F}_{\mathrm{HM}}$ and $\dim M = 30p^5$. It has scJt $[p-1]^{30}[2]^2[1]^{26}$.

Properties of $\mathcal{F}_i$:

(1) $\mathcal{F}_{p-1}(\Omega M) \simeq \mathcal{F}_i(M)(-p+i)$ for $1 \leq i \leq p-1$.
(2) $\mathcal{F}_i(M^*) \simeq \mathcal{F}_i(M)^\vee(-i+1)$ for $1 \leq i \leq p$.
(3) $\mathcal{F}_1(M \otimes_k N) \simeq \bigoplus_{i=1}^{p-1} \mathcal{F}_i(M) \otimes \mathcal{F}_i(N)(i-1)$.
(4) Applying $\mathcal{F}_p$ to $0 \to \Omega M \to P_M \to M \to 0$ yields a sequence

$$0 \to \mathcal{F}_p(\Omega M) \to \mathcal{F}_p(P_M) \to \mathcal{F}_p(M) \to 0$$

that is exact except in the middle, where the homology is $\bigoplus_{i=1}^{p-1} \mathcal{F}_i(M)(-p+i)$.

Theorem 21 (BP 2008)**.** *Given a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$, there exists a kE-module M such that*

(1) *If $p = 2$, then $\mathcal{F}_1(M) = \mathcal{F}$.*
(2) *If $p > 2$, then $\mathcal{F}_1(M) = F * \mathcal{F}$, where $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map.*