THE HOMOLOGICAL CONJECTURES (AFTER HOCHSTER AND ROBERTS)

SIMON SCHIEDER

Homological methods were introduced into commutative algebra in the 1960s, and the homological conjectures are left over from that period.

All rings are commutative and noetherian.

Definition 1. Let $(R, \mathfrak{m})$ be a local ring. Let $k = R/\mathfrak{m}$. *Equal characteristic* means that $\operatorname{char} R = \operatorname{char} k$: i.e. (p, p) or $(0, 0)$. *Mixed characteristic* means that $\operatorname{char} R \neq \operatorname{char} k$: i.e., (p^a, p) for some $a > 1$, or $(0, p)$.

Conjecture 2 (Direct summand conjecture (DSC))**.** Let R be a regular local ring (rlr). Let $R \hookrightarrow S$ be a module-finite R-algebra. Then $R \hookrightarrow S$ splits in the category R-Mod; i.e., R is a direct summand of S.

Conjecture 3 (Monomial conjecture (MC))**.** Let $(R, \mathfrak{m})$ be a local ring of dimension d. Let $x_1, \ldots, x_d$ be a system of parameters. Then for all $t \geq 0$,

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1}).$$

Conjecture 4 (Variant of monomial conjecture (MC'))**.** Let R be a rlr. Let $R \hookrightarrow S$ be a module-finite R-algebra. Then for any system of parameters $x_1, \ldots, x_d$ for R. For all $t \geq 0$,

$$(x_1 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})S.$$

Note: MC, MC', and DSC are equivalent.

One can often reduce to the case that R is complete, by using that the completion $\hat{R}$ is faithfully flat over R. This can be used to prove that MC implies MC'.

MC, MC', DSC are proved in equicharacteristic p, and it follows that they hold also in equicharacteristic 0.

Recall:

Theorem 5 ("vanishing")**.** *Let $(R, \mathfrak{m})$ be a local ring. Let M be a finitely generated R-module. Let $d = \dim M := \dim(R/\operatorname{Ann} M)$. Then $H^d_{\mathfrak{m}}(M) \neq 0$.*

Note: $H^d_{\mathfrak{m}}(R) = \varinjlim_t H^d(\underline{x}^t)$, where $\underline{x} = x_1, \ldots, x_d$ and $\underline{x}^t = x_1^t, \ldots, x_d^t$. The Koszul complex $K^{\cdot}(x)$ for one element is simply

$$0 \to R \xrightarrow{x} R \to 0.$$

There is a map of complexes $K^{\cdot}(x^t) \to K^{\cdot}(x^{t+1})$ given by (id, x). In general, $K^{\cdot}(\underline{x}^t) = K^{\cdot}(x_1^t) \otimes \cdots \otimes K^{\cdot}(x_d^t)$. One has a map of complexes $K^{\cdot}(\underline{x}^t) \to K^{\cdot}(\underline{x}^{t+1})$.

$H^d(\underline{x}^t) = R/(\underline{x}^t)$ maps to $H^d(\underline{x}^{t+1}) = R/(\underline{x}^{t+1})$ via $\phi_{t,i}$. Since $0 \neq \varinjlim H^d(\underline{x}^t)$, there exists t_0 such that for all $t > t_0$ and all i, the map $\phi_{t,i}$ is nonzero. There exists t_0 such that for all $t > t_0$ and all $i \geq 0$, we have $(x_1 \cdots x_d)^i \in (x_1^{t+i}, \ldots, x_d^{t+i})$ (*). Assume that there exists

Date: May 13, 2008.

$k \geq 0$ such that $(x_1 \cdots x_d)^k \in (x_1^{k+1}, \ldots, x_d^{k+1})$. Then $(x_1 \cdots x_d)^{kp^e} \in (x_1^{kp^e+p^e}, \ldots, x_d^{kp^e+p^e})$. Pick e big enough that $p^e > t_0$. This contradicts (*).

Definition 6. Let $(R, \mathfrak{m})$ be a local ring. Let M be an R-module. Then M is a *big CM-module* if and only if $\mathfrak{m}M \neq M$ and there exists a system of parameters for R that is an M-regular sequence. Say that M is a *balanced CM-module* if the same holds for all systems of parameters for M. Say that M is a *small CM-module* if and only if $M \neq 0$ and M is finitely generated and M is balanced.

Recall: *Cohen-Macaulay (CM)* means $\operatorname{depth}_{\mathfrak{m}} M = \dim M$. *Maximal CM* means that $\operatorname{depth}_{\mathfrak{m}} M = \dim R$.

Example 7. Let $R = k[[x, y]]$. Let $M = R \oplus \operatorname{Frac}(R/(y))$. Then x, y is a regular sequence, but y, x is not a regular sequence.

Theorem 8 (Hochster)**.** *If $(R, \mathfrak{m})$ is a local ring containing a field, then there exists a big CM-module over R.*

A local ring R is excellent if and only if

(1) R is universally catenary
(2) For all primes $\mathfrak{p} \in \operatorname{Spec} R$ and all $q \in \operatorname{Spec} R_{\mathfrak{p}}$ and all finite field extensions L over $k(q)$, the tensor product $L \otimes \hat{R}_p$ is regular.
(3) For all finitely generated algebra S over R, the singular locus $\operatorname{Sing} S$ is closed in $\operatorname{Spec} S$.

David Berlekamp's talk in this seminar a few weeks ago proved

Theorem 9. *Let $(R, \mathfrak{m})$ be a local ring. Suppose that R is an excellent domain of characteristic p. Let R^+ be the integral closure of R in an algebraic closure of* $\operatorname{Frac}(R)$. *Then R^+ is a big CM-algebra.*

Why do we want big CM-modules? If there exists a big CM-module for $(R, \mathfrak{m})$, then MC holds for $(R, \mathfrak{m})$.

Proof. If $x_1, \ldots, x_d$ is a system of parameters, then $\underline{x}$ is a regular M-sequence. Then $\operatorname{gr}_{(\underline{x})} M \simeq M/\underline{x}M[x_1, \ldots, x_d] = M \otimes R/(\underline{x})[x_1, \ldots, x_d]$. □

Definition 10. Let $(R, \mathfrak{m})$ be a local ring. Let $I \subset R$ be an ideal. Then $I^+ := (IR^+) \cap R$ is called the "plus closure".

What properties should a closure $()^{\mathrm{cl}}$ have?

(1) I^{cl} should be an ideal.
(2) $(I^{\mathrm{cl}})^{\mathrm{cl}} = I^{\mathrm{cl}}$.
(3) If $I \subset J$, then $I^{\mathrm{cl}} \subset J^{\mathrm{cl}}$.
(4) $I^+ \subset I^{\mathrm{cl}} \subset \overline{I}$.
(5) *tightness*: If R is a rlr, then $I^{\mathrm{cl}} = I$.
(6) If I is n-generated, then $\overline{I^n} \subset I^{\mathrm{cl}}$.

Theorem 11. *The following are equivalent:*

(1) *Tightness of* $()^+$
(2) *MC'*

(3) *DSC.*

Suppose that $x \in R$. Then $x \in I^+$ if and only if there exists a module-finite $R \hookrightarrow S$ such that $IS \cap R$ contains x.

Suppose that $R \hookrightarrow S$ splits; say $S = R \oplus B$. Then $IS = (IR \oplus IB) \cap (R \oplus 0) = I$.