ALMOST COHEN-MACAULAY ALGEBRAS

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Abstract. One of the classical topics in commutative algebra is the study of Cohen-Macaulay rings and algebras. In addition to their intrinsic interest, they have important applications to other questions, such as the Homological Conjectures. A few years ago Heitmann showed that a weaker condition, which we call the property of being almost Cohen-Macaulay, has many of the same implications and is much more likely to hold. In this talk I will define almost Cohen-Macaulay rings, give several examples, and discuss some of the implications of their existence.

1. Cohen-Macaulay rings

Let R be a local Noetherian ring. Call a sequence of elements $x_1, \ldots, x_d$ a *regular sequence* if $ax_i \in (x_1, \ldots, x_{i-1})$ implies $a \in (x_1, \ldots, x_{n-1})$ for all $a \in R$ and $i = 1, \ldots, d$. Call R *Cohen-Macaulay* if a system of parameters is a regular sequence.

Equivalent definitions:

- For a system of parameters $x_1, \ldots, x_d$, the Koszul complex $K.(x_1, \ldots, x_d)$ is acyclic: $H^i(K.) = 0$ for $i \geq 1$.
- The local cohomology $H^i_{(x_j)}(R)$ (with respect to a system of parameters (x_j)) is 0 for $i = 0, \ldots, d-1$. This is the cohomology of

$$0 \to R \to \prod R_{x_i} \to \prod R_{x_i x_j} \to \cdots \to R_{x_1 \cdots x_d} \to 0.$$

2. The Monomial Conjecture

The "Monomial Conjecture" (Hochster): If $x_1, \ldots, x_d$ are a system of parameters in a local ring R, then $x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})$.

This is true if R is Cohen-Macaulay (because then R is flat over a polynomial ring). This is true if R is equicharacteristic, but open in mixed characteristic for dim ≥ 4.

Almost regular sequences.

Almost zero modules: Let $\mathcal{C}$ be a class of modules over a ring R. We assume

(1) For $0 \to M' \to M \to M'' \to 0$ exact, we have $M \in \mathcal{C}$ if and only if $M', M'' \in \mathcal{C}$.
(2) $\mathcal{C}$ is closed under direct limits.

Example 2.1. $\mathcal{C} = \{0\}$.

Example 2.2 (Gabber-Ramero)**.** "Almost ring theory" (origin: Faltings, "Almost étale extensions"). Fix an ideal $\mathfrak{m}$ with the property $\mathfrak{m}^2 = \mathfrak{m}$. (Faltings: $\mathfrak{m}$ is the ideal generated by $\{p^{1/n} : n \geq 1\}$.) Then M is said to be *almost zero* if $\mathfrak{m}M = 0$.

The condition $\mathfrak{m}^2 = \mathfrak{m}$ is rarely satisfied in Noetherian rings, outside of trivial examples like (0) in a field k, or an ideal in $k \times k$.

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Let R_0 be a complete local Noetherian domain of mixed characteristic p. Let R_0^+ be the absolute integral closure of R_0, that is, the integral closure of R_0 in the algebraic closure of its quotient field. Note that R_0^+ is a union of Noetherian rings, each integral over the ground ring. A typical ring R will be a ring between R_0 and R_0^+.

Define $x_1, \ldots, x_d \in R$ to be a *system of parameters* if it is for some finite extension of R_0.

Say that R is *almost Cohen-Macaulay* (with respect to $\mathcal{C}$) if one of the following definitions holds (we are not sure if they are equivalent):

(1) $\{a : ax_i \in (x_1, \ldots, x_{i-1})\}/(x_1, \ldots, x_{i-1})$ is almost zero (i.e., in $\mathcal{C}$) for a system of a parameters $x_1, \ldots, x_d$.
(2) $H^i_{(x_1,\ldots,x_d)}(R)$ is almost zero for $i = 0, \ldots, d-1$.

Condition (1) implies condition (2).

Theorem 2.3 (Heitmann 2002). *If R_0 is a local domain of mixed characteristic and dimension 3 and if p, x, y is a system of parameters for R_0, then $ap^N \in (x, y)$ implies $ap^{1/n} \in (x, y)R_0^+$ for all $n \geq 1$. This implies the monomial conjecture in this case.*

Open: whether R_0^+ is Cohen-Macaulay.

Theorem 2.4. *Let R be as above (between R_0 and R_0^+), $x_1, \ldots, x_d$ a system of parameters. Let $I = \bigcup_t \{a : ax_1^t \cdots x_d^t \in (x_1^{t+1}, \ldots, x_d^{t+1})$. If $H^i_{(x_1,\ldots,x_d)}(R)$ is almost zero for $i = 0, \ldots, d-1$, then $I/(x_1, \ldots, x_d)$ is almost zero.*

Heitmann's theorem implies: $H^2_{(p,x,y)}(R_0^+)$ is almost zero, annihilated by the ideal $\mathfrak{m}$ generated by $\{p^{1/n} : n \geq 1\}$.

$I/(x, y, p)$ is annihilated by $p^{1/n}$. If the monomial conjecture is false, $1 \in I$, so $p^{1/n} \in (x, y, p)$. Take a valuation v on R_0^+ with $v(R_0^+) \geq 0$; i.e., $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \min\{v(a), v(b)\}$. Then $v(ax + by + cp) \geq \min\{v(x), v(y), v(p)\}$.

Fix a valuation v on R_0^+ with $v(R_0^+) \geq 0$. Let $\mathcal{C}$ be the class of modules M such that for all $m \in M$ and for all $\epsilon > 0$, there exists $r \in R$ such that $v(r) < \epsilon$ and $rm = 0$.

Question 2.5. Is R_0^+ almost Cohen-Macaulay?

(1) (Hochster-Huneke, Huneke-Lyubeznik) In positive characteristic, R_0^+ is Cohen-Macaulay. (In characteristic 0, when the dimension is greater than 3, R_0^+ is not Cohen-Macaulay.)
(2) (R., Singh, Srinivas) In characteristic 0, if R_0 is graded and normal, then the image of $H^2_{\mathfrak{m}}(R_0)$ in $H^2_{\mathfrak{m}}(R_0^+)$ is almost zero. This uses the map of $\operatorname{Proj} R_0$ to its Albanese variety.
(3) Mixed characteristic? Fontaine's theory?

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