ALMOST COHEN-MACAULAY ALGEBRAS

PAUL ROBERTS

Abstract. One of the classical topics in commutative algebra is the study of Cohen-Macaulay rings and algebras. In addition to their intrinsic interest, they have important applications to other questions, such as the Homological Conjectures. A few years ago Heitmann showed that a weaker condition, which we call the property of being almost Cohen-Macaulay, has many of the same implications and is much more likely to hold. In this talk I will define almost Cohen-Macaulay rings, give several examples, and discuss some of the implications of their existence.

1. Cohen-Macaulay rings

Let $R$ be a local Noetherian ring. Call a sequence of elements $x_1, \ldots, x_d$ a regular sequence if $ax_i \in (x_1, \ldots, x_{i-1})$ implies $a \in (x_1, \ldots, x_{n-1})$ for all $a \in R$ and $i = 1, \ldots, d$. Call $R$ Cohen-Macaulay if a system of parameters is a regular sequence.

Equivalent definitions:

- For a system of parameters $x_1, \ldots, x_d$, the Koszul complex $K(x_1, \ldots, x_d)$ is acyclic: $H^i(K) = 0$ for $i \geq 1$.
- The local cohomology $H^i_{(x_j)}(R)$ (with respect to a system of parameters $(x_j)$) is 0 for $i = 0, \ldots, d - 1$. This is the cohomology of

$$0 \to R \to \prod R_{x_i} \to \prod R_{x_ix_j} \to \cdots \to R_{x_1 \cdots x_d} \to 0.$$ 

2. The Monomial Conjecture

The “Monomial Conjecture” (Hochster): If $x_1, \ldots, x_d$ are a system of parameters in a local ring $R$, then $x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})$.

This is true if $R$ is Cohen-Macaulay (because then $R$ is flat over a polynomial ring). This is true if $R$ is equicharacteristic, but open in mixed characteristic for $\dim \geq 4$.

Almost regular sequences.

Almost zero modules: Let $\mathcal{C}$ be a class of modules over a ring $R$. We assume

(1) For $0 \to M' \to M \to M'' \to 0$ exact, we have $M \in \mathcal{C}$ if and only if $M', M'' \in \mathcal{C}$.
(2) $\mathcal{C}$ is closed under direct limits.

Example 2.1. $\mathcal{C} = \{0\}$.

Example 2.2 (Gabber-Ramero). “Almost ring theory” (origin: Faltings, “Almost étale extensions”). Fix an ideal $\mathfrak{m}$ with the property $\mathfrak{m}^2 = \mathfrak{m}$. (Faltings: $\mathfrak{m}$ is the ideal generated by $\{p^{1/n} : n \geq 1\}$.) Then $M$ is said to be almost zero if $\mathfrak{m}M = 0$.

The condition $\mathfrak{m}^2 = \mathfrak{m}$ is rarely satisfied in Noetherian rings, outside of trivial examples like $(0)$ in a field $k$, or an ideal in $k \times k$. 

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Let $R_0$ be a complete local Noetherian domain of mixed characteristic $p$. Let $R_0^+$ be the absolute integral closure of $R_0$, that is, the integral closure of $R_0$ in the algebraic closure of its quotient field. Note that $R_0^+$ is a union of Noetherian rings, each integral over the ground ring. A typical ring $R$ will be a ring between $R_0$ and $R_0^+$.

Define $x_1, \ldots, x_d \in R$ to be a system of parameters if it is for some finite extension of $R_0$.

Say that $R$ is almost Cohen-Macaulay (with respect to $C$) if one of the following definitions holds (we are not sure if they are equivalent):

1. $\{a : ax_i \in (x_1, \ldots, x_{i-1})\}/(x_1, \ldots, x_{i-1})$ is almost zero (i.e., in $C$) for a system of parameters $x_1, \ldots, x_d$.

2. $H^i_{(x_1, \ldots, x_d)}(R)$ is almost zero for $i = 0, \ldots, d - 1$.

Condition (1) implies condition (2).

**Theorem 2.3** (Heitmann 2002). If $R_0$ is a local domain of mixed characteristic and dimension 3 and if $p, x, y$ is a system of parameters for $R_0$, then $ap^n \in (x, y)$ implies $ap^{1/n} \in (x, y)R_0^+$ for all $n \geq 1$. This implies the monomial conjecture in this case.

Open: whether $R_0^+$ is Cohen-Macaulay.

**Theorem 2.4.** Let $R$ be as above (between $R_0$ and $R_0^+$), $x_1, \ldots, x_d$ a system of parameters. Let $I = \bigcup \{a : ax^t \in (x_1, \ldots, x_{i-1})\}$. If $H^i_{(x_1, \ldots, x_d)}(R)$ is almost zero for $i = 0, \ldots, d - 1$, then $I/(x_1, \ldots, x_d)$ is almost zero.

Heitmann’s theorem implies: $H^2_{(p, x, y)}(R_0^+)$ is almost zero, annihilated by the ideal $m$ generated by $\{p^{1/n} : n \geq 1\}$.

$I/(x, y, p)$ is annihilated by $p^{1/n}$. If the monomial conjecture is false, $1 \in I$, so $p^{1/n} \in (x, y, p)$. Take a valuation $\nu$ on $R_0^+$ with $\nu(R_0^+) \geq 0$; i.e., $\nu(ab) = \nu(a) + \nu(b)$ and $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$. Then $\nu(ax + by + cp) \geq \min\{\nu(x), \nu(y), \nu(p)\}$.

Fix a valuation $\nu$ on $R_0^+$ with $\nu(R_0^+) \geq 0$. Let $C$ be the class of modules $M$ such that for all $m \in M$ and for all $\epsilon > 0$, there exists $r \in R$ such that $\nu(r) < \epsilon$ and $rm = 0$.

**Question 2.5.** Is $R_0^+$ almost Cohen-Macaulay?

1. (Hochster-Huneke, Huneke-Lyubeznik) In positive characteristic, $R_0^+$ is Cohen-Macaulay. (In characteristic 0, when the dimension is greater than 3, $R_0^+$ is not Cohen-Macaulay.)

2. (R., Singh, Srinivas) In characteristic 0, if $R_0$ is graded and normal, then the image of $H^2_m(R_0)$ in $H^2_m(R_0^+)$ is almost zero. This uses the map of Proj $R_0$ to its Albanese variety.

3. Mixed characteristic? Fontaine’s theory?

University of Utah